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Distant Coordinates in Matrix Form*

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In the Euclidean plane: given three non-colinear points (vertices of the "triangle of reference") any point is uniquely determined by its distances to those vertices. These are called the "distance coordinates" of the point. The main result of our first paper was to determine which vectors of three non-negative numbers could be distance coordinates for the given reference triangle. In this paper we put that result, and others, into matrix form. This leads to generalizations, and to the effect of a change of the reference triangle on the distance coordinates and the formulas in which they are involved.

Key words: Barycentric coordinates; distance coordinates; Euclidean plane; matrices of order three; triangle of reference.

In [1] [1] we defined distance coordinates for a point in the Euclidean plane with respect to a triangle of reference. We exhibited formulas relating these coordinates to the area coordinates of the point.

In this paper we put those formulas into matrix form, note how they are effected by a change in the triangle of reference, and generalize them considerably. The results are listed in section III, labeled (R.1) through (R.30).

Sections I and II state our definitions and notations, and contain equations (1) through (27). Sections IV and V contain our proofs and equations (28) through (63).

I. Matrix Definitions

Unless stated otherwise capital Roman letters will denote matrices of order three. For example I is the identity matrix, and J is the matrix with 1 in each position. Small Greek letters will denote one by three vectors. For example

$$\epsilon=(1\ 1\ 1).$$

Given a matrix $X=(x_{ij})$ its transpose will be denoted by $X^T=(x_{ji})$. Its determinant is det X. Its adjoint is

$$X^A=\begin{pmatrix} x_{22}x_{33}-x_{23}x_{32} & x_{13}x_{32}-x_{33}x_{12} & x_{12}x_{23}-x_{22}x_{13} \\ x_{23}x_{31}-x_{33}x_{21} & x_{11}x_{33}-x_{13}x_{31} & x_{21}x_{13}-x_{11}x_{23} \\ x_{32}x_{21}-x_{22}x_{31} & x_{31}x_{12}-x_{11}x_{32} & x_{11}x_{22}-x_{12}x_{21} \end{pmatrix} \tag{1}$$

Of course

$$XX^A=(\det X)I=X^AX. \tag{2}$$

The sum of the elements in X^A is

$$h(X)=x_{11}x_{22}+x_{11}x_{33}+x_{22}x_{33}-x_{12}x_{21}-x_{13}x_{31}-x_{23}x_{32}-x_{11}(x_{23}+x_{32})-x_{22}(x_{13}+x_{31})-x_{33}(x_{12}+x_{21}) \\ +x_{12}x_{23}+x_{13}x_{32}+x_{21}x_{13}+x_{23}+x_{31}x_{31}x_{12}+x_{32}x_{21} \tag{3}$$

Note that $(X^A)^T=(X^T)^A$ and that $h(X^T)=h(X)$.

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[1] Figures in brackets indicate the literature references at the end of this paper.

Given a vector $\mu=(m_1\ m_2\ m_3)$ we define the matrix

$$S_\mu=\begin{pmatrix} 0 & m_3 & -m_2 \\ -m_3 & 0 & m_1 \\ m_2 & -m_1 & 0 \end{pmatrix}$$

It has the properties

$$\det S_\mu=0,\ \mu S_\mu=0,\ S_\mu{}^T=-S_\mu$$

and

$$S_\mu{}^A=\mu^T\mu.$$

Also

$$\nu S_\mu=-\mu S_\nu \tag{4}$$

for any vector ν. The special notation S is used for S_ϵ

$$S=\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

We have

$$S^A=\epsilon^T\epsilon=J. \tag{5}$$

Throughout this paper we shall assume the following results which we shall prove in the last section.

Given a matrix X and vectors μ, ν we have

$$\det (X+\mu^T\nu)=\det X+\nu X^A\mu^T \tag{6}$$

and

$$(X+\mu^T\nu)^A=X^A+S_\nu X^T S_\mu{}^T. \tag{7}$$

Given matrices X, Y, Z, with $h(X)\neq 0$, such that

$$h(X)Z=JX^A+YSX^TS^T, \tag{8}$$

then

$$\det Z=h(Y)/h(X). \tag{9}$$

If $h(Y)\neq 0$ then

$$h(Y)Z^{-1}=JY^A+XSY^TS^T. \tag{10}$$

Finally for any matrix X:

$$XSX^TS^T=h(X)I-JX^A, \tag{11}$$

or, equivalently,

$$SX^TS^TX=h(X)I-X^AJ. \tag{12}$$

II. Geometric Definitions

Let t be a triangle in the Euclidean plane with area $\Delta_t>0$ and vertices p_1,p_2,p_3. With repect to t (the *triangle of reference*) a point p has unique area and distance coordinates:

$$p=(b_1,b_2,b_3)=[d_1,d_2,d_3]$$

respectively. The b_i are normalized (i.e. $b_1+b_2+b_3=1$) barycentric coordinates, and

$$d_i=|p-p_i| \qquad i=1, 2, 3$$

is the distance from p to the vertex p_i. Define the vectors

$$\beta_{p/t}=(b_1b_2b_3)$$

and

$$\delta_{p/t}=(d_1^2d_2^2d_3^2).$$

For example for the vertices of t we have

$$\begin{aligned}\beta_{p_1/t}&=(1\quad 0\quad 0)\\ \beta_{p_2/t}&=(0\quad 1\quad 0)\\ \beta_{p_3/t}&=(0\quad 0\quad 1)\end{aligned} \tag{13}$$

and

$$\begin{aligned}\delta_{p_1/t}&=(0\quad a_3^2\quad a_2^2)\\ \delta_{p_2/t}&=(a_3^2\quad 0\quad a_1^2)\\ \delta_{p_3/t}&=(a_2^2\quad a_1^2\quad 0)\end{aligned} \tag{14}$$

where a_i is the length of the side of t opposite the vertex p_i, $i=1, 2, 3$.

Let t_1 be a triangle with area Δ_{t_1} and vertices q_1, q_2, q_3. Define the matrices

$$\mathrm{B}_{t_1/t}=\begin{bmatrix}\beta_{q_1/t}\\ \beta_{q_2/t},\\ \beta_{q_3/t}\end{bmatrix} \quad \mathrm{D}_{t_1/t}=\begin{bmatrix}\delta_{q_1/t}\\ \delta_{q_2/t}\\ \delta_{q_3/t}\end{bmatrix}$$

An equivalent definition for $\mathrm{D}_{t_1/t}$ is

$$\mathrm{D}_{t_1/t}=(|q_i-p_j|^2), \tag{15}$$

so that

$$\mathrm{D}_{t/t_1}=\mathrm{D}_{t_1/t}^{\mathrm{T}}. \tag{16}$$

For example, equations (13) tell us that

$$\mathrm{B}_{t/t}=\mathrm{I},$$

and equations (14) tell us that

$$\mathrm{D}_{t/t}=\begin{bmatrix}0 & a_3^2 & a_2^2\\ a_3^2 & 0 & a_1^2\\ a_2^2 & a_1^2 & 0\end{bmatrix}. \tag{17}$$

We note that

$$\det \mathrm{D}_{t/t}=2a_1^2a_2^2a_3^2, \tag{18}$$

that

$$\mathrm{D}^{\mathrm{A}}_{t/t}=\begin{pmatrix} -a_1^4 & a_1^2a_2^2 & a_1^2a_3^2 \\ a_1^2a_2^2 & -a_2^4 & a_2^2a_3^2 \\ a_1^2a_3^2 & a_2^2a_3^2 & -a_3^4 \end{pmatrix}; \tag{19}$$

that

$$\epsilon\, \mathrm{D}^{\mathrm{A}}_{t/t}=(a_1^2c_1 \quad a_2^2c_2 \quad a_3^2c_3) \tag{20}$$

where

$$c_i=a_1^2+a_2^2+a_3^2-2a_i^2, \qquad i=1, 2, 3$$

and that

$$h(\mathrm{D}_{t/t})\equiv\epsilon\mathrm{D}^{\mathrm{A}}_{t/t}\epsilon^{\mathrm{T}}=16\Delta_t^2. \tag{21}$$

It is also important to note that

$$\mathrm{SD}_{t/t}\mathrm{S}^{\mathrm{T}}=\begin{pmatrix} -2a_1^2 & c_3 & c_2 \\ c_3 & -2a_2^2 & c_1 \\ c_2 & c_1 & -2a_3^2 \end{pmatrix} \tag{22}$$

For any point p define $g_{p/t}$ to be the negative of the power of p with respect to the circumcircle of t. That is, $g_{p/t}$ is the square of the circumradius of t minus the square of the distance from p to the circumcenter of t. Define the vector

$$\gamma_{t_1/t}=(g_{q_1/t}g_{q_2/t}g_{q_3/t}). \tag{23}$$

Note that $g_{p/t}=0$ if and only if p is on the circumcircle of t. Since each vertex of t is on its circumcircle, it follows that $\gamma_{t/t}=0$.

To the results listed in the next section should be added the following which are well known (see [2], pages 218-219).

$$|\det \mathrm{B}_{t_1/t}|=\Delta_{t_1}/\Delta_t. \tag{24}$$

Thus $\mathrm{B}_{t_1/t}$ is non-singular if and only if $\Delta_{t_1}>0$. In that case

$$\mathrm{B}_{t/t_1}=\mathrm{B}^{-1}_{t_1/t}, \tag{25}$$

and

$$\mathrm{B}_{p/t_1}=\mathrm{B}_{p/t}\,\mathrm{B}_{t/t_1} \tag{26}$$

for any point p. Moreover, if t_2 is any triangle with $\Delta_{t_2}>0$ then

$$\mathrm{B}_{t_1/t}\,\mathrm{B}_{t/t_2}=\mathrm{B}_{t_1/t_2}. \tag{27}$$

Finally, if β is any real one by three vector such that $\beta\epsilon^{\mathrm{T}}=1$ then there is a point p such that $\beta=\beta_{p/t}$.

III. List of Results

If δ is any real, non-negative one by three vector, then there is a point p such that $\delta=\delta_{p/t}$ if and only if

(R.1) $$\det (D_{t/t}-\epsilon^{T}\delta-\delta^{T}\epsilon)=0.$$

If $\Delta_{t_1}=0$ then $D_{t_1/t}$ is singular (as is $B_{t_1/t}$). But

(R.2) if $\Delta_{t_1/t}>0$ then $D_{t_1/t}$ is singular if and only if the circumcircles of t and t_1 are orthogonal;

i.e. the circles intersect and the respective radii to a point of intersection are perpendicular.

For any two points p, q we have

(R.3) $$|p-q|^2=(\delta_{p/t}-\delta_{q/t})\beta^{T}_{q/t}.$$

Thus

(R.4) $$(\delta_{p/t}-\delta_{q/t})\ (\beta_{p/t}+\beta_{q/t})^{T}=0.$$

Also

(R.5) $$\beta_{p/t_1}D_{t_1/t}\beta^{T}_{q/t}=|p-q|^2+g_{p/t_1}+g_{q/t} \qquad \Delta_{t_1}>0.$$

In some cases we may desire a formula to be independent of the dimensions of t, or independent of the distance or area coordinates. The following three equations show that this can be done for $g_{p/t}$:

(R.6) $$g_{p/t}=\delta_{p/t}\beta^{T}_{p/t}$$

(R.7) $$g_{p/t}=\frac{1}{2}\beta_{p/t}D_{t/t}\beta^{T}_{p/t}$$

(R.8) $$g_{p/t}=(16\Delta_t^2)^{-1}\ (\det D_{t/t}-\delta_{p/t}D^{A}_{t/t}\epsilon^{T}).$$

We can replace $\delta_{p/t}$ in a formula by using

(R.9) $$\delta_{p/t}=\beta_{p/t}D_{t/t}-g_{p/t}\epsilon.$$

And we can replace $\beta_{p/t}$ in a formula by using

(R.10) $$\beta_{p/t}=(16\Delta_t^2)^{-1}(\epsilon D^{A}_{t/t}+\delta_{p/t}SD_{t/t}S^{T}).$$

If we wish to change our triangle of reference from t to t_1, it turns out that we require $h(D_{t_1/t})\neq 0$. The following formulas show that this is equivalent to $\Delta_{t_1}>0$:

(R.11) $$h(D_{t_1/t})=16\Delta_t^2 \det B_{t_1/t}$$

(R.12) $$|h(D_{t_1/t})|=16\Delta_t\Delta_{t_1}.$$

The first three "change of basis" formulas are

(R.13) $$\delta_{p/t_1}=\beta_{p/t}D^{T}_{t_1/t}-g_{p/t}\epsilon$$

(R.14) $$h(D_{t_1/t})\beta_{p/t_1}=\epsilon D^{A}_{t_1/t}+\delta_{p/t}S\ D^{T}_{t_1/t}S^{T} \qquad \Delta_{t_1}>0$$

(R.15) $$h(D_{t_1/t})g_{p/t_1}=\det D_{t_1/t}-\delta_{p/t}D^{A}_{t_1/t}\epsilon^{T} \qquad \Delta_{t_1}>0.$$

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Note that the last two equations depend only on distance coordinates. (For reliance on area coordinates alone equations (25) and (26) tell us that $\beta_{p/t_1}=\beta_{p/t}\,B_{t_1/t}^{-1}$ if $\Delta_{t_1}>0$.)

These three equations can be rewritten

(R.16) $$\delta_{p/t_1}=\beta_{p/t}\,(D_{t_1/t}-\epsilon^T\delta_{p/t})^T$$

(R.17) $$h(D_{t_1/t})\,\beta_{p/t_1}=\epsilon(D_{t_1/t}-\epsilon^T\delta_{p/t})^A \qquad \Delta_{t_1}>0$$

(R.18) $$h(D_{t_1/t})\,g_{p/t_1}=\det\,(D_{t_1/t}-\epsilon^T\delta_{p/t}) \qquad \Delta_{t_1}>0.$$

We note that $h(D_{t_1/t}-\epsilon^T\delta_{p/t})=h(D_{t_1/t})$. Also

(R. 19) $$\det\,(D_{t_1/t}-\epsilon^T\delta_{p/t}-\delta_{p/t_1}^T\,\epsilon)=0.$$

Other "change of basis" formulas are

(R. 20) $$\delta_{p/t_1}-\delta_{p/t}=\beta_{p/t}\,(D_{t_1/t}-D_{t/t})^T$$

(R. 21) $$D_{t_1/t_1}-D_{t_1/t}=B_{t_1/t}\,(D_{t_1/t}-D_{t/t})^T$$

(R. 22) $$D_{t_1/t_1}=B_{t_1/t}\,D_{t_1/t}^T-\gamma_{t_1/t}^T\epsilon$$

and

(R. 23) $$h(D_{t_1/t})\,B_{t/t_1}=J\,D_{t_1/t}^A+D_{t/t}\,S\,D_{t_1/t}^T\,S^T.$$

Recall that $D_{t/t_1}=D_{t_1/t}^T$ (equation (16)), and that $B_{t/t_1}=B_{t_1/t}^{-1}$ if $\Delta_{t_1}>0$ (equation (25)).

The similarities between many of these equations is due to the fact that they are special cases of the following formulas.

Let t_1, t_2, t_3, t_4 be any triangles. Then

(R.24) $$B_{t_1/t_2}\,(D_{t_2/t_3}-D_{t_2/t_4})=D_{t_1/t_3}-D_{t_1/t_4} \qquad \Delta_{t_2}>0$$

(R.25) $$h(D_{t_2/t_3})B_{t_1/t_2}=J\,D_{t_2/t_3}^A+D_{t_1/t_3}S\,D_{t_2/t_3}^T\,S^T \qquad \Delta_{t_2}\Delta_{t_3}>0$$

(R.26) $$h(D_{t_3/t_4})\,(B_{t_1/t_3}-B_{t_2/t_3})=(D_{t_1/t_4}-D_{t_2/t_4})S\,D_{t_3/t_4}^T S^T \qquad \Delta_{t_3}\Delta_{t_4}>0.$$

Also

(R.27) $$B_{t_1/t_2}D_{t_2/t_3}=D_{t_1/t_3}+\gamma_{t_1/t_2}^T\epsilon \qquad \Delta_{t_2}>0$$

(R.28) $$B_{t_1/t_2}\gamma_{t_2/t_3}^T=\gamma_{t_1/t_3}^T-\gamma_{t_1/t_2}^T \qquad \Delta_{t_2}\Delta_{t_3}>0$$

(R.29) $$B_{t_1/t_2}\gamma_{t_2/t_1}^T=-\gamma_{t_1/t_2}^T \qquad \Delta_{t_1}\Delta_{t_2}>0$$

and

(R.30) $$B_{t_1/t_2}D_{t_2/t_3}B_{t_4/t_3}^T=D_{t_1/t_4}+\gamma_{t_1/t_2}^T\epsilon+\epsilon^T\gamma_{t_4/t_3} \qquad \Delta_{t_2}\Delta_{t_3}>0.$$

For example (R.5) is a special case of (R.30) as follows:

Reduce triangle t_1 to a point p and triangle t_4 to a point q. Then $B_{t_1/t_2}=\epsilon^T\beta_{p/t_2}$, $B_{t_4/t_3}=\epsilon^T\beta_{q/t_3}$, $D_{t_1/t_4}=|p-q|^2\,J$, $\gamma_{t_1/t_2}=g_{p/t_2}\epsilon$ and $\gamma_{t_4/t_3}=g_{p/t_3}\epsilon$. All terms reduce to multiples of $J=\epsilon^T\epsilon$ whose equality is (R.5) with $t_2\to t_1$ and $t_3\to t$.

IV. Proofs

We will refer to several equations in [1] and will use the notation "(12′)" for "(12) in [1]", or "equation (12) in [1]."

Equation (12′) can be written

$$32\Delta_t^2 f(x_1, x_2, x_3)=2\sum_{k=1}^{3} a_k^2 c_k x_k-2a_1^2a_2^2a_3^2-2\sum_{k=1}^{3} a^2{}_k x_k^2+2c_1x_2x_3+2c_2x_1x_3+2c_3x_1x_2.$$

Let $\delta=(x_1x_2x_3)$ and look at equations (20), (18), and (22). Then we have

$$32\Delta_t^2 f(x_1, x_2, x_3)=2\epsilon D^{\mathrm{A}}_{t/t}\delta^{\mathrm{T}}-\det D_{t/t}+\delta \mathrm{S}\, D_{t/t}\mathrm{S}^{\mathrm{T}}\delta^{\mathrm{T}} \tag{28}$$

We shall prove that the r.h.s. is $-\det(D_{t/t}-\epsilon^{\mathrm{T}}\delta-\delta^{\mathrm{T}}\epsilon)$. Since the Theorem in [1] states that (assuming $\delta\geq 0$) there is a point p such that $\delta=\delta_{p/t}$ if and only if $f(x_1, x_2, x_3)=0$, we will have proved (R.1).

From equation (6) we have

$$\begin{aligned}\det(D_{t/t}-\epsilon^{\mathrm{T}}\delta-\delta^{\mathrm{T}}\epsilon)&=\det(D_{t/t}-\epsilon^{\mathrm{T}}\delta)-\epsilon(D_{t/t}-\epsilon^{\mathrm{T}}\delta)^{\mathrm{A}}\delta^{\mathrm{T}}\\ &=\det D_{t/t}-\delta D^{\mathrm{A}}_{t/t}\epsilon^{\mathrm{T}}-\epsilon(D_{t/t}-\epsilon^{\mathrm{T}}\delta)^{\mathrm{A}}\epsilon^{\mathrm{T}}.\end{aligned} \tag{29}$$

From (7) we have

$$\begin{aligned}\epsilon(D_{t/t}-\epsilon^{\mathrm{T}}\delta)^{\mathrm{A}}&=\epsilon D^{\mathrm{A}}_{t/t}-\epsilon S_\delta D^{\mathrm{T}}_{t/t}S^{\mathrm{T}}\\ &=\epsilon D^{\mathrm{A}}_{t/t}+\delta S D_{t/t}S^{\mathrm{T}}\end{aligned}$$

because $D^{\mathrm{T}}_{t/t}=D_{t/t}$ and $\epsilon S_\delta=-\delta S$ (see (4)). Substituting into (29) yields

$$\det(D_{t/t}-\epsilon^{\mathrm{T}}\delta-\delta^{\mathrm{T}}\epsilon)=\det D_{t/t}-\delta D^{\mathrm{A}}_{t/t}\epsilon^{\mathrm{T}}-\epsilon D^{\mathrm{A}}_{t/t}\delta^{\mathrm{T}}-\delta S D_{t/t}S^{\mathrm{T}}\delta^{\mathrm{T}}.$$

Since $\delta D^{\mathrm{A}}_{t/t}\epsilon^{\mathrm{T}}=(\epsilon D^{\mathrm{A}}_{t/t}\delta^{\mathrm{T}})^{\mathrm{T}}$, the two scalars are equal. Comparison with the r.h.s. of (28) proves our result and thus (R. 1). We shall prove (R. 2) at the end of this section.

Given points p, q let $\beta_{p/t}=(b_1b_2b_3)$ and $\beta_{q/t}=(b_1'b_2'b_3')$.
From (28′) we have

$$2|p-q|^2=\sum_{k=1}^{3} c_k(b_k-b_k')^2 \tag{30}$$

From (33′) we have

$$2g_{p/t}=\sum_{k=1}^{3} c_k(b_k-b_k^2)\,. \tag{31}$$

Thus

$$2|p-q|^2+2g_{p/t}+2g_{q/t}=\sum_{k=1}^{3} c_k(b_k+b_k'-2b_kb_k') \tag{32}$$

Using $c_k=a_1^2+a_2^2+a_3^2-2a_k^2$, $k=1, 2, 3$, this becomes

$$|p-q|^2+g_{p/t}+g_{q/t}=a_1^2(b_2b_3'+b_3b_2')+a_2^2(b_1b_3'+b_3b_1')+a_3^2(b_1b_2'+b_2b_1'). \tag{33}$$

The r.h.s is just $\beta_{p/t}D_{t/t}\beta^{\mathrm{T}}_{q/t}$ so that

$$|p-q|^2=\beta_{p/t}D_{t/t}\beta^{\mathrm{T}}_{q/t}-g_{p/t}-g_{q/t}. \tag{34}$$

We proceed to show that $\beta_{p/t}D_{t/t}\beta^{T}_{q/t}=g_{p/t}+\delta_{p/t}\beta^{T}_{q/t}$. We have $\beta_{p/t}D_{t/t}=(b_2a_3^2+b_3a_2^2\ldots\ \ldots)$. From (43′) this becomes $\beta_{p/t}D_{t/t}=(d_1^2+g_{p/t}\ldots\ \ldots)$ or

$$\beta_{p/t}D_{t/t}=\delta_{p/t}+g_{p/t}\epsilon. \tag{35}$$

Since $\epsilon\beta^{T}_{q/t}=1$ we have

$$\beta_{p/t}D_{t/t}\beta^{T}_{q/t}=\delta_{p/t}\beta^{T}_{q/t}+g_{p/t}. \tag{36}$$

Substituting this into (34) we get

$$|p-q|^2=\delta_{p/t}\beta^{T}_{q/t}-g_{q/t}. \tag{37}$$

From (37′) we get

$$g_{p/t}=\delta_{p/t}\beta^{T}_{p/t}. \tag{38}$$

Then (37) becomes $|p-q|^2=\delta_{p/t}\,\beta^{T}_{q/t}-\delta_{q/t}\,\beta^{T}_{q/t}$, and (R. 3) is proved. Interchanging p and q yields $|p-q|^2=(\delta_{q/t}-\delta_{p/t})\,\beta^{T}_{p/t}$. Equating the two r.h.s. proves (R. 4). And (R. 5) was proved by the remarks at the end of the last section (once (R. 30) is proved).

We have already proved (R. 6) with (38). Setting $q=p$ in (36) and using (38) proves (R. 7). Multiply (35) on the right by $D^{A}_{t/t}\,\epsilon^{T}$ to get $\det D_{t/t}=\delta_{p/t}\,D^{A}_{t/t}\,\epsilon^{T}+h(D_{t/t})\,g_{p/t}$. Using (21) completes the proof of (R. 8).

We have already proved (R. 9) with (35). Equations (16′) can be written

$$16\Delta_t^2\,b_1=a_1^2\,c_1-2a_1^2\,d_1^2+c_3\,d_2^2+c_2\,d_3^2\qquad\text{etc.} \tag{39}$$

Equations (20) and (22) translate this into

$$16\Delta_t^2\,\beta_{p/t}=\epsilon\,D^{A}_{t/t}+\delta_{p/t}\,SD_{t/t}\,S^{T} \tag{40}$$

which is (R. 10).

Setting $p=q_1,\ q_2,\ q_3$ in (40) yields

$$16\Delta_t^2\,B_{t_1/t}=JD^{A}_{t_1/t}+D_{t_1/t}\,SD_{t/t}\,S^{T}. \tag{41}$$

Since $16\Delta_t^2=h\,(D_{t/t})$ and $D_{t/t}=D^{T}_{t/t}$ we can apply the results following equation (8) with $X=D_{t/t}$, $Y=D_{t_1/t}$ and $Z=B_{t_1/t}$. Equation (9) proves (R. 11) and equation (10) proves (R. 23) since $B_{t_1/t}=B^{-1}_{t_1/t}$ if $\Delta_{t_1}>0$. Result (R. 12) follows from (R. 11) and (24).

Interchange p and q in equation (37) and rewrite it as $|p-q|^2=\beta_{p/t}\delta^{T}_{q/t}-g_{p/t}$. Letting $q=q_1$, q_2, q_3 yields (R. 13). Multiply (R. 23) on the left by $\beta_{p/t}$. Since $\beta_{p/t}B_{t/t_1}=\beta_{p/t_1}$, $\beta_{p/t}J=\epsilon$, $\beta_{p/t}D_{t/t}=\delta_{p/t}+g_{p/t}\epsilon$, and $\epsilon S=0$ we get (R. 14). Result (R. 15) follows from (R. 13) and (R. 14) with the formula $g_{p/t_1}=\beta_{p/t_1}\delta^{T}_{p/t_1}$. In the calculation use $S^{T}\epsilon^{T}=0$ and $SD^{T}_{t_1/t}S^{T}D_{t_1/t}=h(D_{t_1/t})I-D^{A}_{t_1/t}J$ (from (12).)

Result (R. 16) is (R. 13) rewritten to conform with the two following results. They in turn follow from (R. 14) and (R. 15) by using the same kind of calculations as those surrounding equation (29). Multiply (R. 17) on the right by ϵ^{T}, note that $\beta_{p/t_1}\epsilon^{T}=1$ and that the r.h.s. is $h(D_{t_1/t}-\epsilon^{T}\delta_{p/t})$ to show that the latter equals $h(D_{t_1/t})$. To prove (R. 19) write (R. 13) as

$$\beta_{p/t}(D_{t_1/t}-\epsilon^{T}\delta_{p/t}-\delta^{T}_{p/t_1}\epsilon)^{T}=0.$$

Since $\beta_{p/t}$ cannot be the zero vector (because $\beta_{p/t}\epsilon^{T}=1$) it follows that the matrix $D_{t_1/t}-\epsilon^{T}\delta_{p/t}-\delta^{T}_{p/t_1}\epsilon$ cannot be non-singular.

Result (R. 20) is (R. 13) minus (R. 9). Then $p=q_1,\ q_2,\ q_3$, in (R. 20) proves (R. 21). Similarly (R. 22) comes from (R. 13) by setting $p=q_1,\ q_2,\ q_3$, and recalling the definition (23).

Let $t_1=t_3$ in (R. 13) and then set $p=q_1, q_2, q_3$, to get

$$D_{t_1/t_3}=B_{t_1/t}D^T_{t_3/t}-\gamma^T_{t_1/t}\epsilon$$

with $t=t_2$ (thus requiring $\Delta_{t_2}>0$) and $D^T_{t_3/t_2}=D_{t_2/t_3}$ we get (R. 27). In that equation set $t_3=t_4$ and subtract from the original to get (R. 24). Let $t_1=t_2$ in (R. 14) and then $p=q_1, q_2, q_3$. This yields (R. 25) with $t=t_3$.

Result (R. 26) comes from the following observation: Let $X=D_{t_2/t_2}$ in equation (11) to get

$$JD^A_{t_2/t_3}=h(D_{t_2/t_3})I-D_{t_2/t_3}SD^T_{t_2/t_3}S^T.$$

Substituting this into (R. 25) yields

$$h(D_{t_2/t_3})(B_{t_1/t_2}-I)=(D_{t_1/t_3}-D_{t_2/t_3})SD^T_{t_2/t_3}S^T. \tag{42}$$

Subtract this from the same expression with $t_1=t_4$ to get

$$h(D_{t_2/t_3})(B_{t_4/t_2}-B_{t_1/t_2})=(D_{t_4/t_3}-D_{t_1/t_3})SD^T_{t_2/t_3}S^T.$$

The cyclic permutation (1 2 3 4) transforms this into (R. 26).

Use the cyclic permutation (1 2 3) on (R. 27) to get

$$B_{t_2/t_3}D_{t_3/t_1}=D_{t_2/t_1}+\gamma^T_{t_2/t_3}\epsilon \qquad \Delta_{t_3}>0.$$

Assume $\Delta_{t_2}>0$ and multiply on the left by B_{t_4/t_2}. Using (27) we get

$$B_{t_4/t_3}D_{t_3/t_1}=B_{t_4/t_2}D_{t_2/t_1}+B_{t_4/t_2}\gamma^T_{t_2/t_3}\epsilon.$$

Now $B_{t_4/t_3}D_{t_3/t_1}=D_{t_4/t_1}+\gamma^T_{t_4/t_3}\epsilon$ and $B_{t_1/t_2}D_{t_2/t_1}=D_{t_4/t_1}+\gamma^T_{t_4/t_2}\epsilon$ (using (R. 27) with the proper permutations). Thus

$$\gamma^T_{t_4/t_3}\epsilon=\gamma^T_{t_4/t_2}\epsilon+B_{t_4/t_2}\gamma^T_{t_2/t_3}\epsilon.$$

With $t_4=t_1$ the first column of this matrix equation is (R. 28). In (R. 28) set $t_3=t_1$ and recall that $\gamma_{t_1/t_2}=0$ to get (R. 29).

Assume $\Delta_{t_3}>0$ and multiply (R. 27) on the right by $B^T_{t_4/t_3}$ to get

$$B_{t_1/t_2}D_{t_2/t_3}B^T_{t_4/t_3}=D_{t_1/t_3}B^T_{t_4/t_3}+\gamma^T_{t_1/t_2}\epsilon \qquad \Delta_{t_2}\Delta_{t_3}>0 \tag{43}$$

Using (R. 27) we have $B_{t_4/t_3}D_{t_3/t_1}=D_{t_4/t_1}+\gamma^T_{t_4/t_3}\epsilon$. Transposed this is $D_{t_1/t_3}B^T_{t_4/t_3}=D_{t_1/t_4}+\epsilon^T\gamma_{t_4/t_3}$. Substitution into (43) yields (R. 30).

We finish this section by proving (R.2) in the course of which we will use capital Roman letters which do not denote matrices. Specifically R_t and O_t will denote the circumradius and the circumcenter respectively of the triangle t. The distance from O_t to each vertex of t is R_t so that

$$\delta_{0_t/t}=R_t^2\epsilon. \tag{44}$$

Letting $p=O_t$ in (R.15) yields

$$\det D_{t_1/t}=h(D_{t_1/t})\ (g_{0_t/t}+R_t^2) \qquad \Delta_{t_1}>0.$$

By definition, $g_{p/t_1}=R_{t_1}^2-|p-O_{t_1}|^2$. Thus

$$\det D_{t_1/t}=h(D_{t_1/t})\ (R_t^2+R_{t_1}^2-|O_t-O_{t_1}|^2) \qquad \Delta_{t_1}>0. \tag{45}$$

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Since $R_t^2+R_{t_1}^2=|O_t-O_{t_1}|^2$ if and only if the respective circumcircles are orthogonal, we have proved (R.2).

V. Matrix Proofs

Let $X=(x_{ij})$ be a matrix of order three, let $\nu=(n_1n_2n_3)$, and let

$$\sigma=\nu X^A=(s_1s_2s_3).$$

Since $h(Q)=\epsilon Q^A\epsilon^T$, $\epsilon^T\epsilon=J$, and $S_\epsilon=S$, we see that letting $\omega=\mu=\epsilon$ in equations (60), (61), and (63) yield equations (8), (9), and (10) respectively (with $Y=P$, $X=Q$). Similarly, letting $\mu=\nu=\epsilon$ in equations (51) and (52) yield equations (11) and (12) resp ctively. Finally, equation (55) is equation (6), and equation (54) is equation (7).

By inspection of (1) we have

(46) $$s_1=x_{21}(n_3x_{32}-n_2x_{33})+x_{22}(n_1x_{33}-n_3x_{31})+x_{23}(n_2x_{31}-n_1x_{32})$$
$$=-x_{31}(n_3x_{22}-n_2x_{23})-x_{32}(n_1x_{23}-n_3x_{21})-x_{33}(n_2x_{21}-n_1x_{22})$$

(47) $$s_2=x_{31}(n_3x_{12}-n_2x_{13})+x_{32}(n_1x_{13}-n_3x_{11})+x_{33}(n_2x_{11}-n_1x_{12})$$
$$=-x_{11}(n_3x_{32}-n_2x_{33})-x_{12}(n_1x_{33}-n_3x_{31})-x_{13}(n_2x_{31}-n_1x_{32})$$

(48) $$s_3=x_{11}(n_3x_{22}-n_2x_{23})+x_{12}(n_1x_{23}-n_3x_{21})+x_{13}(n_2x_{21}-n_1x_{22})$$
$$=-x_{21}(n_3x_{12}-n_2x_{13})-x_{22}(n_1x_{13}-n_3x_{11})-x_{23}(n_2x_{11}-n_1x_{12})$$

Let γ_1, γ_2, γ_3 denote the rows of XS_ν^T, where

$$S_\nu^T=\begin{pmatrix}0 & -n_3 & n_2\\ n_3 & 0 & -n_1\\ -n_2 & n_1 & 0\end{pmatrix}$$

We have

$$\gamma_1=(n_3x_{12}-n_2x_{13}\quad n_1x_{13}-n_3x_{11}\quad n_2x_{11}-n_1x_{12})$$
$$\gamma_2=(n_3x_{22}-n_2x_{23}\quad n_1x_{23}-n_3x_{21}\quad n_2x_{21}-n_1x_{22})$$
$$\gamma_3=(n_3x_{32}-n_2x_{33}\quad n_1x_{33}-n_3x_{31}\quad n_2x_{31}-n_1x_{32})$$

Let ρ_1, ρ_2, ρ_3 denote the rows of X:

$$\rho_i=(x_{i1}x_{i2}x_{i3})\qquad i=1,\ 2,\ 3.$$

Clearly

$$\rho_i\gamma_i^T=0\qquad i=1,\ 2,\ 3.$$

From equations (46), (47), (48) we see that

$$s_1=\rho_2\gamma_3^T=-\rho_3\gamma_2^T$$
$$s_2=\rho_3\gamma_1^T=-\rho_1\gamma_3^T$$
$$s_3=\rho_1\gamma_2^T=-\rho_2\gamma_1^T.$$

Thus $X(XS_\nu^T)^T=(\rho_i\gamma_j^T)$ equals

$$\begin{pmatrix}0 & s_3 & -s_2\\ -s_3 & 0 & s_1\\ s_2 & -s_1 & 0\end{pmatrix}=S_\sigma.$$

That is

(49) $$XS_\nu X^T=S_\sigma\equiv S_{\nu X^A}.$$

Let μ be a one by three vector. Since

$$S_\sigma S_\mu{}^T=(\sigma\mu^T)I-\mu^T\sigma \tag{50}$$

(a general result), we have

$$XS_\nu X^T S_\mu{}^T=(\nu X^A\mu^T)I-\mu^T\nu X^A. \tag{51}$$

Taking transposes and substituting X^T for X yields

$$S_\mu X^T S_\mu{}^T X=(\nu X^A\mu^T)I-X^A\nu^T\mu. \tag{52}$$

Let

$$W=X^A+S_\nu X^T S_\mu{}^T$$

Then

$$(X+\mu^T\nu)W=(\det X)\ I+XS_\nu X^T S_\mu{}^T+\mu^T\nu X^A.$$

since $XX^A=(\det X)I$ and $\nu S_\nu=0$. Using (51) yields

$$(X+\mu^T\nu)W=(\det X+\nu X^A\mu^T)I. \tag{53}$$

Since this is an identity in the variables X, μ, ν (or rather, their elements) we conclude that W is a scalar times $(X+\mu^T\nu)^A$. In the latter the elements of order two in the x_{ij} form X^A, as is true of W, so that the scalar is 1. Thus $W=(X+\mu^T\nu)^A$:

$$(X+\mu^T\nu)^A=X^A+S_\nu X^T S_\mu{}^T, \tag{54}$$

and from (53):

$$\det (X+\mu^T\nu)=\det X+\nu X^A\mu^T. \tag{55}$$

Let P, Q be matrices of order three, and let ω be a one by three vector. In equations (54) and (55) we will set $X=PS_\omega Q^T S_\mu{}^T$ and $\nu=\omega Q^A$. Since $S_\mu{}^A=\mu^T\mu$ we have

$$\begin{aligned}(PS_\omega Q^T S_\mu{}^T)^A&=(S_\mu{}^T)^A(Q^T)^A S_\omega{}^A P^A\\ &=\mu^T\mu(Q^A)^T\omega^T\omega P^A\\ &=(\omega Q^A\mu^T)\mu^T\omega P^A.\end{aligned} \tag{56}$$

(The scalars $\mu(Q^A)^T\omega^T$ and $\omega Q^A\mu^T$ are equal). From (49) we have $S_{\omega Q^A}=QS_\omega Q^T$. Thus

$$(PS_\omega Q^T S_\mu{}^T+\mu^T\omega Q^A)^A=(\omega Q^A\mu^T)\mu^T\omega P^A+(QS_\omega Q^T)(S_\mu QS_\omega{}^T P^T)S_\mu{}^T. \tag{57}$$

From (51) we have $Q^T S_\mu QS_\omega{}^T=(\omega Q^A\mu^T)I-\omega^T\mu(Q^A)^T$. Since $S_\omega\omega^T=0$, (57) becomes

$$(PS_\omega Q^T S_\mu{}^T+\mu^T\omega Q^A)^A=(\omega Q^A\mu^T)(\mu^T\omega P^A+QS_\omega P^T S_\mu{}^T). \tag{58}$$

In using (54) we recall that $\det S_\mu=0$. Thus

$$\begin{aligned}\det (PS_\omega Q^T S_\mu{}^T+\mu^T\omega Q^A)&=(\omega Q^A)(\omega Q^A\mu^T)\mu^T\omega P^A\mu^T\\ &=(\omega Q^A\mu^T)^2\omega P^A\mu^T.\end{aligned} \tag{59}$$

Suppose $\omega Q^A\mu^T$ is not 0, and define the matrix Z by

$$(\omega Q^A\mu^T)Z=PS_\omega Q^T S_\mu{}^T+\mu^T\omega Q^A. \tag{60}$$

From (59) we have $(\omega Q^A\mu^T)^3\det Z=(\omega Q^A\mu^T)^2(\omega P^A\mu^T)$
or

$$\det Z=(\omega P^A\mu^T)/(\omega Q^A\mu^T). \tag{61}$$

The 1.h.s. of (58) is $(\omega Q^A \mu^T)^2 Z^A$ so that

$$(\omega Q^A \mu^T) Z^A = Q S_\omega P^T S_\mu{}^T + \mu^T \omega P^A. \tag{62}$$

Assume $\omega P^A \mu^T \neq 0$. Then $Z^A = (\det Z) Z^{-1}$ so that

$$(\omega P^A \mu^T) Z^{-1} = Q S_\omega P^T S_\mu{}^T + \mu^T \omega P^A. \tag{63}$$

Comparing with (60) we note that the transformation $P \leftrightarrow Q$ takes Z into Z^{-1}.

VI. References

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